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THE USE OF THE MOMENT METHOD TO DERIVE THE GAS AND PLASMA TRANSPORT EQUATIONS WITH TRANSPORT COEFFICIENTS IN HIGHER-ORDER APPROXIMATIONS†

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Grad's moment method is used to derive the linear equations of mass momentum and energy transfer of the components and to obtain all the transport coefficients (kinetic coefficients) for a multicomponent mixture of monatomic gases. A system of equations for the expansion coefficients of the non-equilibrium correction to the distribution function using a system of irreducible tensorial Hermite polynomials (the equations of moments) is obtained on the basis of the linearized Boltzmann equations for the components of the mixture. The assumptions under which these equations reduce to a system of algebraic equations for determining of the mass diffusion fluxes, the heat fluxes of the components and the partial viscous stress tensors are analysed. The advantage in writing the transport equation in a "forces in terms of fluxes" representation for solving actual problems of the flows of multicomponent mixtures as compared with the classical "fluxes in terms of forces" representation in the standard Chapman-Enskog method [1–3] is demonstrated. Different ways of representing the transport equations and expressions for the transport coefficients are considered in an arbitrary order of approximation with respect to the number of Sonine polynomials which are retained in the expansion of the distribution function (the Chapman–Cowling method). This enables one, in particular, to establish a direct link between results obtained by different methods and to track more clearly the constraints which are actually used when employing classical method [1–3] and the modified method [4, 5] of deriving the transport equations and calculating the transport coefficients in the Chapman–Enskog scheme. © 2003 Elsevier Ltd. All rights reserved.

1. INTRODUCTION

The formal kinetic theory of monatomic [1–3] and polyatomic [6–8] gas mixtures is based on the use of systems of kinetic equations with collision integrals in the Boltzmann or Wang Chang–Uhlenbeck form. In the case of low Knudsen numbers, these equations are solved using the traditional Chapman–Enskog method (CEM) [1–3] or Grad's moment method (GMM) [9–11], which enable one to obtain both the equations for the conservation (balance) of mass, momentum and energy for mixtures of gases (the equations of multicomponent hydrodynamics) as well as the linear transport equations for the diffusion fluxes of the components of the mixture, the heat flux and the stress tensor for a mixture which close them.

In the usual CEM scheme [1–3], the linear transport equations are written in a form which is solved for the diffusion fluxes and the heat flux (for a mixture) in terms of the gradients of the molar fractions (concentrations) of the components, the pressure gradients and the temperatures, and the differences in the mass external forces acting on the different components of the mixture. Here, the stress tensor depends linearly on the rate of deformation tensor and the divergence of the mean mass velocity (the shear and bulk viscosity). We shall henceforth denote this way of writing the transport equations by the term "fluxes in terms of forces". All of the coefficients occurring in this, it can be said, "classical' form of representation of the transport equations, that is, the multicomponent coefficients of diffusion and thermal diffusion and the "instantaneous" (non-true) thermal conductivity λ', are described in the form of ratios of determinants of order $N\xi + 1$ to determinants of order $N\xi$ which are obtained as the result of solving a truncated infinite system of algebraic equations using Kramer's ruler. Here, N is the number of components in the mixture and ξ is the order of the approximations (the number of terms which are retained in finding the transport coefficients in the form of series in orthogonal Sonine polynomials in the Chapman – Cowling method (CCM) [1]. In the solutions obtained, the elements of the determinants are expressed in terms of so-called bracket integrals (integral brackets) of Sonine polynomials of different orders which, in their turn, are represented in the form of linear combinations of a set (indices l and s) $\Omega^{ls}_{\alpha\beta}(T)$ of integrals which are functions of temperature and also depend on the parameters of the interaction potentials between particles of species α and β [1–3].

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In the case of mixtures of monatomic gases formed from electrically neutral particles, a rapid convergence of the above-mentioned series is observed, and it is therefore sufficient to take just a small number of terms of the expansion into account in this method in order to obtain reasonable accuracy when calculating the transport coefficients. Taking account of just the lower-order approximations at temperatures from room temperature up to a temperature below the temperature of the dissociation of diatomic molecules (2000–3000 K) gives an error which, as a rule, does not exceed 0.3% for the viscosity coefficient, 0.5% for the thermal conductivity and 10% for the thermal diffusion coefficient [2, 3]. We note that, according to existing terminology [3], retention of the first non-zero coefficients in the expansion for the transport coefficients – this is the first approximation ($\xi = 1$) in the case of the viscosity and diffusion coefficients and the second approximation ($\xi = 2$) in the case of the "instantaneous" thermal conductivity λ' and thermal diffusion coefficients, corresponds to the so-called lowest approximations. Thermal diffusion is therefore sometimes referred to as a second-order effect [2, 3].

Taking account of the higher approximations when calculating the transport coefficients becomes of fundamental importance in the case of partially or fully ionized gas mixtures [12–15]. We note that methods for solving the kinetic equations, developed for neutral gases, can also be used completely successfully in the case of a plasma if the latter is considered as a multicomponent mixture of neutral and charged particles and the divergence which arises in the effective (integrated with respect to the scattering angle) collision cross-sections of the charged particles is removed using a screened Coulomb potential or using the operation of a formal cutoff of the impact parameter at a distance of the order of the Debye radius [10–12, 16]. Certain mathematical difficulties arise when there is a magnetic field. However, these can be completely surmounted using a simple generalization of the CEM [1, 3] or the GMM [10, 11]. The numerical convergence of the transport coefficients of a plasma, obtained using well-known methods for solving kinetic equations, has been investigated in many papers [12–17, 18].

It should be noted that the expressions for the transport coefficients, obtained from the solution of linear integral equations for perturbed parts of the distribution function using the CEM both by a variational method [3] as well as by direct expansion in series in Sonine polynomials (CCM [1]), are found to be completely identical. Since these equations are self-adjoint, the variational method give a monotonically decreasing or monotonically increasing sequence of values of the transport coefficients (with the exception, perhaps, of the thermal diffusion coefficient). It follows from this that, in each higher approximation, a more accurate value is obtained than in the preceding approximation and that no oscillations occur.

Numerous transport coefficient calculations, which have been carried out up to the present time, show that the rate of convergence of the expansions is different for the different transport coefficients and depends on the nature of the behaviour of the potential function of the interacting particles (the "curvature" of the potential, for example) and on the ratio of the masses of the components (in particular, on the existence of a light component in the mixture). In the case of a fully ionized plasma, the third approximation [13, 15, 20] gives values of the transport coefficients which are close to the exact values [19] while, in the case of weakly ionized gases, the order of approximation in the case of the electron transport coefficients which ensures the required accuracy depends considerably on the nature of the dependence of the electron – atom collision cross-section on the energy of the electron.

For instance, in the case of partially ionized argon, the very pronounced Ramsauer minimum in this dependence, which is observed at low electron energies, leads to an appreciable deterioration in the convergence of the approximations: in the case of intermediate degrees of ionization, for example, at least the sixth approximation is required in order to obtain a more or less accurate value of the electrical conductivity and, in the Lorentz limit, even the twelfth approximation does not ensure the required accuracy [12, 14]. In this connection we note that the formulae for calculating the electron transport coefficients can be appreciably simplified if use is made of the fact that the ratio of the mass of an electron to the mass of heavy particles in small [12, 22]. However, in this case also, calculation of the electron coefficients in the higher approximations remains a laborious problem. Of course, the volume of the calculations increases considerably if the higher approximations for the transport coefficients of the heavy components (atoms, molecules and ions) are considered since, in this case, high-order determinants have to be calculated with elements of complex structure which depend on the ratios of the masses of the components, the concentrations and the temperature.

We will illustrate the complexity of the problems which arise in such calculations by considering problems in hypersonic aerodynamics and heat and mass transfer. The partially ionized air plasma, which is formed behind the main shock wave at the entry velocity of a space vehicle in an atmosphere with a second cosmic velocity of 11.2 km/s or higher, contains (without taking account of argon) up to eleven components (molecules, atoms, ions of the major components of the air mixture and electrons) [23].

In heat and mass transfer problems at these velocities, up to several tens of components are formed [24, 25] due to the blowing in of gaseous products from the evaporation and dissociation of the heat-resistant coatings and their chemical interaction with the products of partially ionized air in the shock layer. A multiply ionized plasma with a large number of components is formed in the shock layer around meteoroids entering the Earth's atmosphere or the atmospheres of other planets. Calculations of the flow in the shock layer of a multi-element, partially ionized gas mixture, taking account of the higher approximations (for example, the second approximation for the coefficient of viscosity and the third approximation for the thermal conductivity and thermal diffusion coefficient) using the classical "fluxes in terms of forces" representation [1–3] are found to be unrealizable in practice and have not been carried out up to the present time. Using this classical scheme, transport coefficients have only been calculated for a simple (single element) plasma [13, 21, 26, 27] and for partially ionized, eight-component air in thermochemical equilibrium under static conditions ($\mathbf{v} = 0$, $\nabla p = 0$) and with a fixed composition of the chemical elements [17, 18]. Meanwhile, in the case of the actual formulation of the problem of the flow of a multicomponent mixture, not only do the temperature and the pressure change but, also, the elemental composition, due to the different diffusion properties of the components and thermal and barodiffusion [28].

Together with the difficulties which have been noted above, there is another important fact which makes it difficult to use expressions written in the "fluxes in terms of forces" representation. Substitution of the "classical" expressions for the diffusion fluxes and the heat flux into the equations for the conservation (balance) of the mass of the components and the energy of the mixture leads to a system of equations which have not been solved for the higher derivatives of the required functions since, in each of them, there will simultaneously be second derivatives of the temperature and of all the concentrations. At the present time, there are no general methods for the efficient numerical solution of such systems of equations even in the approximation of different versions of the Navier – Stokes equations which have been asymptotically simplified with respect to the Reynolds number [29, 30]. In order to obtain a system of multicomponent hydrodynamic equations which is simple and convenient for numerical solution, it is necessary to have transport equations which are solved for the "thermodynamic forces" (which we shall call the "forces in terms of fluxes" representation). Then, together with the transport equations, written in this form, the parabolized Navier – Stokes equations (which have been simplified by discarding, in the continuity and energy equations, the first derivatives of the fluxes with respect to the marching (longitudinal) coordinate, which are of the order of the inverse of the Reynolds number) give a system of equations which are solved for the first derivatives of the concentrations, temperature and, also, the diffusion fluxes and heat flux with respect to the normal to the surface past which the flow occurs, that is, the normal Cauchy form [31], for which efficient numerical methods have been developed [23, 32, 33]. The advantages of the transport equations in the "forces in terms of fluxes" form lies in the fact that this form of writing down the equations is as though specially adapted for calculating all the effective transport coefficients in a finite form in the case of locally. thermochemically equilibrium flows with concentrations of the chemical elements which are changing in the stream [34, 35]. In this case, by taking account of multicomponent diffusion, one can reveal the phenomenon of the separation of chemical elements [28].

Writing down the equations for the mass transport of the components in the form of the so-called Stefan–Maxwell equations, in which the "thermodynamic force of diffusion", which includes the concentration and pressure gradients and the difference in the mass forces of the components, is expressed in terms of a linear combination of the diffusion fluxes and a thermal diffusion term which is proportional to the temperature gradient, corresponds to the "forces in terms of fluxes" representation. The equation for heat transfer in the mixture is written in the same representation, that is, in a form which is solved for the temperature gradient (with the true thermal conductivity λ) in terms of the mass diffusion fluxes and the heat flux. Attempts to obtain these expressions using known "classical" results have only been realized in the lower approximations and, in particular, require the use of the operation of double matrix inversion [36, 37].

It has been shown [4, 5, 38] that the transport equations for a multicomponent partially ionized mixture of gases in the "forces in terms of fluxes" representation can be obtained by a certain modification of the method of solving the infinite system of algebraic equations for the expansion coefficients in the CCM. Here, the first few coefficients are expressed in terms of the diffusion and thermal fluxes which are of interest to us and can be found from the solution of the general system of equations in any specified approximation with respect to the number of Sonine polynomials in the expansion. The corresponding transport coefficients are written in the form of a ratio of determinants of order $N(\xi - 1) + 1$ to determinants of order $N(\xi - 1)$, the elements of which are just integral brackets of Sonine polynomials. As a result, much simpler expressions (without the need for double matrix inversion) were obtained for the true thermal conductivity λ and the thermal diffusion ratios and, thereby, the simplest expression for the heat flux in an arbitrary approximation of the CCM. The Stefan–Maxwell equations, that is, the mass transport equations of the components, taking thermal diffusion and the corrections of the higher approximations to the binary diffusion coefficients into account, were obtained in the same approximation. The transport equations derived in this way were also generalized to the case of a plasma taking an external magnetic field into account [39, 40].

Grad's moment method (GMM) [9–11] is an independent method for deriving the transport equations and for calculating the transport coefficients, and is an alternative to the CEM. It had already been shown in 1962 [41] that the use of this method in the case of a multicomponent gas mixture makes it possible, in the well-known approximation of $13N$ moments, to obtain the mass transport equations of the components in the form of the Stefan–Maxwell equations, that is, in the "forces in terms of fluxes" form and the expression for the heat flux which is written at once with the "true" thermal conductivity. At the same time, the results obtained correspond, as regards the accuracy of the calculation of the kinetic coefficients, to the complete second approximation in the expansion in Sonine polynomials in the CCM method [1–3]. This approach was later generalized to the case of multi-temperature, partially and fully ionized multispecies plasma in the presence of a magnetic field [42–45, 10, 11] and, also, to the case of polyatomic gases and gaseous mixture [7].

At the same time, the GMM can be generalized to the case when a large number of polynomials, and the coefficients corresponding to them, are taken into account in the expansion of the distribution function [10, 11] which enables one to develop a scheme for obtaining expressions for the diffusion fluxes and heat flux in a multicomponent gas mixture in the "forces in terms of fluxes" form with transport coefficients which are calculated in any approximation.

Below, an infinite system of coupled quasilinear differential equations for the expansion coefficients of the distribution function in a system of irreducible tensorial Hermite polynomials (the equations of moments) is obtained on the basis of the linearized kinetic Boltzmann equation. The assumptions under which these equations reduce to a system of algebraic equations for determining the mass diffusion fluxes, the heat fluxes of the components and the partial viscosity stress tensors, which is equivalent to the systems of equations obtained in the modified method developed in [4, 5, 39, 40], are considered. The different forms of representation of the transport equations and the expressions for the transport coefficients in an arbitrary order of approximation are discussed. This enables us, in particular, to establish a direct link between the results obtained using the different independent approaches and to track more clearly the constraints which are actually used in the conventional and modified procedures for solving the system of kinetic Boltzmann equations in the CEM.

2. THE EXPANSION OF THE DISTRIBUTION FUNCTION AND THE EQUATION OF MOMENTS

We will consider an N-component, partially ionized gas mixture made up of an arbitrary number of species of neutral atoms, ions and electrons. The non-equilibrium state of such a mixture (a plasma) is described by a distribution function of particles of species α which we will seek in the form

$$f_\alpha = f_\alpha^{(0)}(1+\phi_\alpha), \quad f_\alpha^{(0)} = n_\alpha\left(\frac{\gamma_\alpha}{2\pi}\right)^{3/2}\exp\left(-\frac{\gamma_\alpha c_\alpha^2}{2}\right) \tag{2.1}$$

where $f_\alpha^{(0)}$ is the local Maxwell distribution, ϕ_α is a small correction ($|\phi_\alpha| \ll 1$), $\gamma_\alpha = m_\alpha/kT$, m_α is the mass of a particle, T is the temperature, k is Boltzmann's constant, n_α is the number density of particles of species α, $\mathbf{c}_\alpha = \mathbf{v}_\alpha - \mathbf{u}$ is the relative velocity of a particle and $\mathbf{u}$ is the mean mass velocity of the mixture.

The correction $\phi_\alpha(\mathbf{v}_\alpha, \mathbf{r}, t)$ satisfies the system of linearized kinetic Boltzmann equations [3, 8]

$$D_\alpha \ln f_\alpha^{(0)} + D_\alpha \phi_\alpha = \sum_\beta L_{\alpha\beta}\phi_\beta \tag{2.2}$$

where the differential operator notation

$$D_\alpha = \frac{d}{dt} + (\mathbf{c}_\alpha \cdot \nabla) + \left(\frac{\mathbf{F}_\alpha}{m_\alpha} \cdot \nabla_v\right), \quad \frac{d}{dt} = \frac{\partial}{\partial t} + (\mathbf{u} \cdot \nabla) \tag{2.3}$$

is used.

Here, $\mathbf{F}_\alpha$ is the external force acting on the particle and ∇_υ is the gradient operator in velocity space. In the general case

$$\mathbf{F}_\alpha = e_\alpha \mathbf{E} + \mathbf{X}_\alpha$$

where $\mathbf{E}$ is the electric field strength, $\mathbf{X}_\alpha$ are forces of a non-electromagnetic nature, and $e_\alpha = Z_\alpha e$ is the charge of a particle (for electrons, $Z_e = -1$). For simplicity, we shall assume that there is no magnetic field present.

The linearized collision operator $L_{\alpha\beta}$ on the right-hand side of Eq. (2.2) is defined in such a way that [3]

$$L_{\alpha\beta}\phi_\beta = \int f_\beta^{(0)}(\phi'_\alpha + \phi'_{1\beta} - \phi_\alpha - \phi_{1\beta}) g \sigma_{\alpha\beta}(d\Omega) d\mathbf{v}_{1\beta} \tag{2.4}$$

Here, g is the relative velocity of the colliding particles, $\sigma_{\alpha\beta}(g, \Omega)$ is the differential scattering cross-section, Ω is the solid angle of scattering, primes refer to quantities which are determined after the collision of the particles, and the subscript 1 is introduced in order to distinguish identical colliding particles when $\alpha = \beta$.

Taking into account the fact that

$$\ln f_\alpha^{(0)} = \ln n_\alpha - \frac{3}{2}\ln T - \frac{1}{2}\gamma_\alpha(\mathbf{v}_\alpha - \mathbf{u})^2 + \text{const} \tag{2.5}$$

the expression for $D_\alpha \ln f_\alpha^{(0)}$ is easily transformed.
As a result,

$$\begin{aligned} D_\alpha \ln f_\alpha^{(0)} = \Bigg\{ & \left(\frac{1}{n_\alpha}\frac{dn_\alpha}{dt} + \nabla \cdot \mathbf{u}\right) + \frac{1}{2}(\gamma_\alpha c_\alpha^2 - 3)\left(\frac{1}{T}\frac{dT}{dt} + \frac{2}{3}\nabla \cdot \mathbf{u}\right) + \\ & + \mathbf{c}_\alpha\left[\frac{1}{p_\alpha}\nabla p_\alpha + \gamma_\alpha\left(\frac{d\mathbf{u}}{dt} - \frac{\mathbf{F}_\alpha}{m_\alpha}\right)\right] + \frac{1}{2}\mathbf{c}_\alpha(\gamma_\alpha c_\alpha^2 - 5)\frac{1}{T}\nabla T + \gamma_\alpha\left(c_{\alpha i}c_{\alpha k} - \frac{1}{3}\delta_{ik}c_\alpha^2\right)\varepsilon_{ik}\Bigg]\Bigg\} \end{aligned} \tag{2.6}$$

Here, $p_\alpha = n_\alpha kT$ and the rate of shear tensor (or the rate of deformation tensor with a trace equal to zero)

$$\varepsilon_{ik} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i}\right) - \frac{1}{3}\delta_{ik}\nabla \cdot \mathbf{u} \tag{2.7}$$

is introduced.

We now expand the non-equilibrium correction ϕ_α in series in an orthogonal system of irreducible† tensorial polynomial H_α^{mn} (ξ_α) of the dimensionless relative velocity of the particles $\xi_\alpha = \gamma_\alpha^{1/2}$ [46, 10, 11]

$$\phi_\alpha = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \sigma_{mn} a_{\alpha i_1 \ldots i_m}^{mn}(\mathbf{r}, t) H_{\alpha i_1 \ldots i_m}^{mn}(\xi_\alpha) \tag{2.8}$$

Here, σ_{mn} is a normalizing factor

$$\sigma_{mn} = \frac{(2m+1)!(m+n)!}{n!(m!)^2(2m+2n+1)!}$$

The subscripts $i_1, \ldots, i_m$ correspond to the Cartesian components of a rank m tensor (henceforth, these subscripts are omitted).

Apart from normalization, the irreducible tensorial Hermite polynomials $H^{mn}(\xi)$ are the product of the Sonine polynomials $S_{m+1/2}^n(\xi^2/2)$ and the tensorial spherical harmonics $P^{(m)}(\xi)$ [6. 46]

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†An irreducible tensor is a tensor with a zero convolution with respect to any pair of indices; in particular, an irreducible second-rank tensor in continuum mechanics is called a deviator; the tensor (2.7) is a deviator.

$$H^{mn}(\boldsymbol{\xi}) = (-2)^n n! S^n_{m+1/2}(\xi^2/2) P^{(m)}(\boldsymbol{\xi}) \tag{2.9}$$

The first few polynomials $P^{(m)}(\xi)$ have the form

$$\begin{gathered} P^{(0)} = 1, \quad P^{(1)}_i = \xi_i, \quad P^{(2)}_{ik} = \xi_i\xi_k - \frac{1}{3}\delta_{ik}\xi^2 \\ P^{(3)}_{ikl} = \xi_i\xi_k\xi_l - \frac{1}{5}\xi^2(\xi_i\delta_{kl} + \xi_k\delta_{il} + \xi_l\delta_{ik}) \end{gathered} \tag{2.10}$$

We now introduce the definition of a scalar product of functions in Hilbert space

$$\langle g_\alpha, h_\alpha \rangle = \frac{1}{n_\alpha}\int f^{(0)}_\alpha g_\alpha h_\alpha d\mathbf{c}_\alpha \tag{2.11}$$

The orthogonality condition for the polynomials $H^{mn}(\xi)$ then takes the form [8]

$$\langle H^{mn}, H^{m'n'} \rangle = (\alpha^{mn})^2 \delta_{mm'}\delta_{nn'}\Delta^{(m)} \tag{2.12}$$

where α^{mn} is a normalizing coefficient:

$$\alpha^{mn} = \left[\frac{2^n m!(2m+2n+1)!!}{n!(2m+1)!!}\right]^{1/2}$$

δ_{ss} is the Kronecker delta, $\Delta^{(m)}$ is a unit projection tensor and, in particular [8]

$$\Delta^{(1)}_{ik} = \delta_{ik}, \quad \Delta^{(2)}_{ikjl} = \frac{1}{2}(\delta_{ij}\delta_{kl} + \delta_{il}\delta_{kj}) - \frac{1}{3}\delta_{ik}\delta_{jl}$$

The coefficients a^{mn}_α in expansion (2.8) are determined from the conditions for the polynomials to be orthogonal by the relation

$$n_\alpha a^{mn}_\alpha = \int H^{mn}_\alpha(\xi_\alpha) f^{(0)}_\alpha \phi_\alpha d\mathbf{v} = n_\alpha \langle H^{mn}_\alpha, \phi_\alpha \rangle \tag{2.13}$$

which enables us to express these coefficients in terms of the corresponding moments of the distribution function. The first few coefficients are written as

$$\begin{gathered} a^{00}_\alpha = 0, \quad a^{01}_\alpha = 3(T_\alpha - T)/T, \quad a^{10}_{\alpha i} = \gamma^{1/2}_\alpha w_{\alpha i} \\ a^{11}_{\alpha i} = 2\gamma^{1/2}_\alpha h_{\alpha i}/p_\alpha, \quad a^{20}_{\alpha ik} = \pi_{\alpha ik}/p_\alpha \end{gathered} \tag{2.14}$$

where

$$\mathbf{h}_\alpha = \mathbf{q}_\alpha - 5/2 p_\alpha \mathbf{w}_\alpha \tag{2.15}$$

is the reduced partial heat flux, while the diffusion velocity $\mathbf{w}_\alpha = \mathbf{u}_\alpha - \mathbf{u}$, the partial viscous stress tensor $\hat{\boldsymbol{\pi}}_\alpha$ and the partial heat flux $\mathbf{q}_\alpha$ are defined by the expressions

$$\mathbf{w}_\alpha = \langle \mathbf{c}_\alpha, \phi_\alpha \rangle, \quad \hat{\boldsymbol{\pi}}_\alpha = \rho_\alpha \langle \overline{\mathbf{c}_\alpha\mathbf{c}_\alpha}, \phi_\alpha \rangle, \quad \mathbf{q}_\alpha = \frac{1}{2}\rho_\alpha \langle c^2_\alpha \mathbf{c}_\alpha, \phi_\alpha \rangle \tag{2.16}$$

Here, $\rho_\alpha = m_\alpha n_\alpha$ is a mass density of particles of species α, $\hat{\boldsymbol{\pi}}_\alpha = \hat{\mathbf{P}}_\alpha - p_\alpha \hat{\boldsymbol{\delta}}$, where $\hat{\mathbf{P}}_\alpha$ is the partial stress tensor, $p_\alpha = n_\alpha kT$ is the partial pressure and $\hat{\boldsymbol{\delta}}$ is a unit second-rank tensor. The temperature of particles of species α is defined here as

$$kT_\alpha = kT + {}^1\!/_3\, m_\alpha \langle c^2_\alpha, \phi_\alpha \rangle \tag{2.17}$$

Notation of the type $\overline{aa}$ ... is used henceforth for irreducible symmetric tensors so that, $(\overline{c_\alpha c_\alpha})_{ik} = c_{\alpha i}c_{\alpha k} - 1/3\delta_{ik}c^2_\alpha$, for example.

The system of linearized equations of the moments is constructed by multiplying the kinetic equation (2.2) by $f_\alpha^{(0)} H_\alpha^{mn}$ with subsequent integration with respect to the velocities. It can be represented in general form as

$$n_\alpha \langle H_\alpha^{mn}, D_\alpha \ln f_\alpha^{(0)} \rangle + n_\alpha \langle H_\alpha^{mn}, D_\alpha \phi_\alpha \rangle = R_\alpha^{mn} \tag{2.18}$$

where

$$R_\alpha^{mn} = \sum_\beta \int H_\alpha^{mn} f_\alpha^{(0)} L_{\alpha\beta} \phi_\beta d\mathbf{c}_\alpha = n_\alpha \langle H_\alpha^{mn}, L\phi_\alpha \rangle, \quad L\phi_\alpha = \sum_\beta L_{\alpha\beta}\phi_\beta \tag{2.19}$$

(R_α^{mn} is a moment with respect to the collision integral).

Note that each of the terms in the expression for $D_\alpha \ln f_\alpha^{(0)}$ contains some of the irreducible Hermite polynomials as a factor since, by definition,

$$\begin{aligned} &H_\alpha^{00} = 1, \quad H_\alpha^{01} = \gamma_\alpha c_\alpha^2 - 3, \quad H_{\alpha i}^{10} = \gamma_\alpha^{1/2} c_{\alpha i} \\ &H_{\alpha i}^{11} = \gamma_\alpha^{1/2} c_{\alpha i}(\gamma_\alpha c_\alpha^2 - 5), \quad H_{\alpha ik}^{20} = \gamma_\alpha (c_{\alpha i} c_{\alpha k} - \delta_{ik} c_\alpha^2) \end{aligned} \tag{2.20}$$

Calculation of the first term on the left-hand side to Eq. (2.18), taking into account the orthogonality condition for the irreducible Hermite polynomials (2.12), gives

$$\begin{aligned} n_\alpha \langle H_\alpha^{mn}, D_\alpha \ln f_\alpha^{(0)} \rangle = n_\alpha \Bigg\{ &\left(\frac{1}{n_\alpha}\frac{dn_\alpha}{dt} + \nabla \cdot \mathbf{u}\right)\delta_{m0}\delta_{n0} + 3\left(\frac{1}{T}\frac{dT}{dt} + \frac{2}{3}\nabla \cdot \mathbf{u}\right)\delta_{m0}\delta_{n1} + \\ &+ \gamma_\alpha^{-1/2}\left[\frac{1}{p_\alpha}\nabla p_\alpha + \gamma_\alpha\left(\frac{d\mathbf{u}}{dt} - \frac{\mathbf{F}_\alpha}{m_\alpha}\right)\right]\delta_{m1}\delta_{n0} + 5\gamma_0^{-1/2}\nabla \ln T \delta_{m1}\delta_{n1} + 2\overline{\nabla \mathbf{u}}\delta_{m2}\delta_{n0} \Bigg\} \end{aligned} \tag{2.21}$$

An irreducible second rank tensor $\nabla \mathbf{u}$ has been introduced here such that $(\nabla \mathbf{u})_{ik} = \varepsilon_{ik}$.

In the expression obtained after appropriate integration in the second term on the left-hand side of Eq. (2.18), all the non-linear terms are omitted, including the product of the moments (or the coefficients $n_\alpha a_\alpha^{mn}$) for small gradients of the corresponding thermodynamic variables and, also, for small potential gradients (weak external fields). As a result, we obtain

$$n_\alpha \langle H_\alpha^{mn}, D_\alpha \phi_\alpha \rangle = \frac{dn_\alpha a_\alpha^{mn}}{dt} + \nabla n_\alpha \langle \mathbf{c}_\alpha H_\alpha^{mn}, \phi_\alpha \rangle + \text{ non-linear terms} \tag{2.22}$$

The second (flow) term on the right-hand side of equality (2.22), after the expansion for ϕ_α (2.8) has been substituted into it, can be represented in the form of a linear combination of the derivatives with respect to a coordinate of the coefficients of $(m + 1)$th and $(m - 1)$th tensor dimensionality [8]

$$\begin{aligned} \nabla n_\alpha \langle \mathbf{c}_\alpha H_\alpha^{mn}, \phi_\alpha \rangle &= \sum_{kl} \sigma_{kl} \langle \mathbf{c}_\alpha H_\alpha^{mn}, H_\alpha^{kl} \rangle \nabla n_\alpha a_\alpha^{kl} = \\ &= \sum_l \sigma_{ml}\left(A_{\alpha mn}^{m+1,l} \nabla n_\alpha a_\alpha^{m+1,l} + B_{\alpha mn}^{m-1,l} \nabla n_\alpha a_\alpha^{m-1,l}\right) \end{aligned} \tag{2.23}$$

The coefficients A_α and B_α are given the expressions

$$A_{\alpha mn}^{m+1,l} = \frac{1}{2m+3}\langle \mathbf{c}_\alpha H_\alpha^{mn}, H_\alpha^{m+1,l} \rangle, \quad B_{\alpha mn}^{m-1,l} = \frac{1}{2m+3}\langle \mathbf{c}_\alpha H_\alpha^{mn}, H_\alpha^{m-1,l} \rangle \tag{2.24}$$

The notation $\nabla n_\alpha a_\alpha^{m-1,l}$ when $m \geqslant 2$ corresponds to a symmetric irreducible tensor. For instance, if $a_\alpha^{m-1,l}$ is a vector (for $m = 2$), then $\nabla n_\alpha a_\alpha^{1,l}$ is an irreducible second-rank tensor with components

$$\frac{1}{2}\left(\frac{\partial n_\alpha a_{\alpha i}^{1l}}{\partial x_j} + \frac{\partial n_\alpha a_{\alpha j}^{1l}}{\partial x_i}\right) - \frac{1}{3}\delta_{ij}\frac{\partial n_\alpha a_{\alpha l}^{1l}}{\partial x_l}$$

We will now calculate the right-hand side of the equations of moments (2.18), that is, the quantities R_α^{mn} (2.19). To do this, we substitute the expressions for the perturbations of the distribution function ϕ_α and ϕ_β into the linearized collision integral using expansion (2.8). As a result, expression (2.19) can be written in the form

$$R_\alpha^{mn} = -\sum_\beta\sum_k\sum_l n_\alpha n_\beta \sigma_{kl}([H^{kl}, H^{mn}]'_{\alpha\beta} a_\alpha^{kl} + [H^{kl}, H^{mn}]''_{\alpha\beta} a_\beta^{kl}) \tag{2.25}$$

were the so-called partial integral brackets of irreducible Hermite polynomials are introduced. The general definition of integral brackets has the form [1–3]

$$\begin{aligned} [F, G]'_{\alpha\beta} &= \frac{1}{n_\alpha n_\beta}\int f_\alpha^{(0)} f_\beta^{(0)}(F_\alpha - F'_\alpha) G_\alpha g \sigma_{\alpha\beta} d\Omega d\mathbf{c}_\alpha d\mathbf{c}_\beta \\ [F, G]''_{\alpha\beta} &= \frac{1}{n_\alpha n_\beta}\int f_\alpha^{(0)} f_\beta^{(0)}(F_\beta - F'_\beta) G_\alpha g \sigma_{\alpha\beta} d\Omega d\mathbf{c}_\alpha d\mathbf{c}_\beta \end{aligned} \tag{2.26}$$

We use the definition of the polynomials $H_\alpha^{mn}(\boldsymbol{\xi}_\alpha)$ (2.9) and change the variable $\boldsymbol{\xi}_\alpha$ to the variable $\mathbf{W}_\alpha$

$$\mathbf{W}_\alpha = \left(\frac{m_\alpha}{2kT}\right)^{1/2}\mathbf{c}_\alpha = 2^{-1/2}\boldsymbol{\xi}_\alpha \tag{2.27}$$

Then,

$$\begin{aligned} H_\alpha^{mn}(\mathbf{W}_\alpha) &= (-1)^n 2^{n+m/2} n! S_{m+1/2}^n(W_\alpha^2) R^{(m)}(\mathbf{W}_\alpha) \\ R^{(m)}(\mathbf{W}_\alpha) &= 2^{-m/2} P^{(m)}(\boldsymbol{\xi}_\alpha) \end{aligned} \tag{2.28}$$

The relation

$$[R^{(k)}, R^{(m)}] = \frac{1}{2m+1}[R^{(m)}, R^{(m)}]\delta_{km}$$

holds [8] for the integral brackets of the polynomials $R^{(m)}(\mathbf{W})$.

The final expression for R_α^{mn} then takes the form

$$R_\alpha^{mn} = -\sum_\beta\sum_l C_{\alpha\beta}^{mnl} a_\beta^{ml} \tag{2.29}$$

where

$$C_{\alpha\beta}^{mnl} = \delta_{\alpha\beta}\sum_\gamma A_{\alpha\gamma}^{mnl} + B_{\alpha\beta}^{mnl} \tag{2.30}$$

and the expressions for $A_{\alpha\beta}$ and $B_{\alpha\beta}$ are written as

$$\begin{aligned} A_{\alpha\beta}^{mnl} &= n_\alpha n_\beta Q_{mnl}[S_{m+1/2}^l(W^2)R^m(\mathbf{W}), S_{m+1/2}^n(W^2)R^{(m)}(\mathbf{W})]'_{\alpha\beta} \\ B_{\alpha\beta}^{mnl} &= n_\alpha n_\beta Q_{mnl}[S_{m+1/2}^l(W^2)R^m(\mathbf{W}), S_{m+1/2}^n(W^2)R^{(m)}(\mathbf{W})]''_{\alpha\beta} \end{aligned} \tag{2.31}$$

The coefficients Q_{mnl} have the form

$$Q_{mnl} = (-2)^{n+l} 2^m \frac{n!l!}{2m+1}\sigma_{ml} = (-1)^{n+l} 2^{n+l+m}\frac{(2m)!(m+l)!n!}{(m!)^2(2m+2l+1)!} \tag{2.32}$$

The integral quantities $[\ldots]'_{\alpha\beta}$ and $[\ldots]''_{\alpha\beta}$ correspond to known partial integral brackets of Sonine polynomials which are introduced in the CEM [1–3].

For the first two tensor polynomials $R^{(m)}(\mathbf{W})$, which are used in writing down the integral brackets in the CEM, we have

$$R_i^{(1)}(\mathbf{W}) = W_i, \quad R_{ik}^{(2)}(\mathbf{W}) = W_iW_k - \delta_{ik}W^2$$

When the expressions obtained above are taken into account, the system of linearized equations for the moments (2.18) can be written in the following final form:

$$\frac{dn_\alpha a_\alpha^{mn}}{dt} + \sum_l \sigma_{ml}(A_{\alpha mn}^{m+1,l}\nabla n_\alpha a_\alpha^{m+1,l} + B_{\alpha mn}^{m-1,l}\nabla n_\alpha a_\alpha^{m-1,l}) + n_\alpha\left\{\left(\frac{1}{n_\alpha}\frac{dn_\alpha}{dt} + \nabla\cdot u\right)\delta_{m0}\delta_{n0} + \right.$$
$$+3\left(\frac{1}{T}\frac{dT}{dt} + \frac{2}{3}\nabla\cdot\mathbf{u}\right)\delta_{m0}\delta_{n1} + \gamma_\alpha^{-1/2}\left[\frac{1}{p_\alpha}\nabla p_\alpha + \gamma_\alpha\left(\frac{d\mathbf{u}}{dt} - \frac{\mathbf{F}_\alpha}{m_\alpha}\right)\right]\delta_{m1}\delta_{n0} + \tag{2.33}$$
$$\left. + 5\gamma_\alpha^{-1/2}\nabla\ln T\delta_{m1}\delta_{n1} + 2\overbrace{\nabla\mathbf{u}}\delta_{m2}\delta_{n0}\right\} = -\sum_\beta\sum_l C_{\alpha\beta}^{mnl}a_\beta^{ml}$$

Equations (2.33) must be supplemented by the relations

$$\sum_\alpha m_\alpha n_\alpha\gamma_\alpha^{-1/2}a_\alpha^{10} = 0, \quad \sum_\alpha n_\alpha a_\alpha^{01} = 0 \tag{2.34}$$

which correspond to the conditions resulting from the definitions of diffusion velocities, the mean mass velocity and the temperature of the mixture,

$$\sum_\alpha m_\alpha n_\alpha \mathbf{w}_\alpha = 0, \quad \sum_\alpha n_\alpha T_\alpha = nT \tag{2.35}$$

The equations of moments (2.33) form an infinite system of coupled equations for scalar ($m = 0$), vector ($m = 1$) and tensor ($m = 2, 3, \ldots$) quantities. A search for specific solutions is possible if it is confined to a finite number of terms in expansion (2.8). Depending on the values taken for m and n, the general system of equations of moments (2.33) decomposes into independent systems of equations for determining the scalar, vector and tensor coefficients $n_\alpha a_\alpha^{mn}$. For instance, when $m = 0$ and $n = 0$, these equations correspond to the equation of continuity, and summation with respect to α of the equations for $m = 1$ and $n = 0$, as well as for $m = 0$ and $n = 1$, leads to the equation of motion and the linearized equation of the energy of the mixture. When conditions (2.35) are taken into account, the corresponding conservation equations take the form

$$\frac{dn_\alpha}{dt} + n_\alpha\nabla\cdot\mathbf{u} + \nabla\cdot n_\alpha\mathbf{w}_\alpha = 0 \tag{2.36}$$

$$\rho\frac{d\mathbf{u}}{dt} + \nabla p + \nabla\hat{\pi} + \sum_\alpha n_\alpha\mathbf{F}_\alpha = 0 \tag{2.37}$$

$$nk\frac{dT}{dt} + \frac{2}{3}p\nabla\cdot\mathbf{u} + \frac{2}{3}\nabla\cdot\mathbf{q} - kT\sum_\alpha\nabla\cdot n_\alpha\mathbf{w}_\alpha = 0 \tag{2.38}$$

Here, $p = nkT$ is the total pressure, $\hat{\pi} = \sum_\alpha \hat{\pi}_\alpha$ is the viscous stress tensor and $\mathbf{q} = \sum_\alpha \mathbf{q}_\alpha$ is the heat flux of the mixture, respectively. Note that, using Eqs (2.37) and (2.38), it is possible to eliminate the time derivatives $d\mathbf{u}/dt$ and dT/dT from the left-hand side of Eq. (2.33) while the equation of continuity (2.36) on the left-hand side of (2.33) is satisfied identically.

3. THE APPROXIMATION OF 13*N* MOMENT

The well known approximation of 13N moment [9, 10] is the most commonly used approximation in the moment method. In this method, terms, including tensorial polynomials of not higher than rank ($m \leq 2$) with values $n = 1$ for $m = 0$, $n = 0$, 1 for $m = 1$ and $n = 0$ for $m = 2$, are retained in the expansion of the distribution function (2.8). The equations of moments are written for the variables n_α (or ρ_α), $\mathbf{u}$ and T, which are included in the weight function (the local Maxwellian distribution) and, also, for the coefficients of the expansion a_α^{01}, a_α^{10}, a_α^{11} and a_α^{20}, which are expressed in terms of moments of the distribution function, which have an explicit physical meaning.

The first few of these moments are the relative temperature difference $(T_\alpha - T)/T$, the diffusion velocity $\mathbf{w}_\alpha$, the partial viscous stress tensor $\pi_{\alpha ik}$ and the reduced partial heat flux $\mathbf{h}_\alpha$ (2.15), which are connected with the corresponding expansion coefficients by relations (2.14).

The expansion of the correction to the distribution function in the approximation of 13N moments takes the form [10]

$$\phi_\alpha = a_{\alpha i}^{10}\xi_{\alpha i} + \frac{1}{6}a_\alpha^{01}(\xi_\alpha^2 - 3) + \frac{1}{2}a_{\alpha ik}^{20}\left(\xi_{\alpha i}\xi_{\alpha k} - \frac{1}{3}\delta_{ik}\xi_\alpha^2\right) + \frac{1}{10}a_{\alpha i}^{11}\xi_{\alpha i}(\xi_\alpha^2 - 5) \tag{3.1}$$

or, in the physical variables (2.14)

$$\phi_\alpha = \gamma_\alpha \mathbf{w}_\alpha \cdot \mathbf{c}_\alpha + \frac{1}{2}\frac{T_\alpha - T}{T}(\gamma_\alpha c_\alpha^2 - 3) + \frac{1}{2}\gamma_\alpha \frac{\pi_{\alpha ik}}{p_\alpha}\left(c_{\alpha i}c_{\alpha k} - \frac{1}{3}\delta_{ik}c^2\right) + \frac{1}{5}\gamma_\alpha \frac{\mathbf{h}_\alpha \cdot \mathbf{c}_\alpha}{p_\alpha}(\gamma_\alpha c_\alpha^2 - 5) \tag{3.2}$$

We now turn to the linearized equations of moments (2.33). It has already been mentioned that known equations for the conservation of the number of particles of species α, and the momentum and energy of the system (2.36)–(2.38) appear in the complete system of equations of moments. When $m = 0$ and $n = 1$, we obtain a system of equations for the relative temperature differences $(T_\alpha - T)/T$. Analysis shows 1[10] that the order of these quantities (when there are no external forces) is determined by the spatial derivatives of the heat flux and the diffusion flux and not by the gradients of the initial macroscopic parameters as in the case of the remaining vector and tensor fluxes. The equations for the temperature differences are therefore not considered below. When $m = 1$ and $n = 0$, 1, a compatible system of equations for the coefficients a_α^{10} and a_α^{11} (or for the diffusion velocities and the reduced heat fluxes of particles of species α) and, when $m = 2$ and $n = 0$, equations for determining the coefficients a_α^{20} (or the partial viscous stress tensors) follow from the equations of moments (2.33).

Instead of the equations for the coefficients $a_{\alpha i}^{10}$, $a_{\alpha ik}^{20}$, and $a_{\alpha i}^{11}$, it is convenient to use the equations for the physical quantities $\mathbf{w}_\alpha$, $\pi_{\alpha ik}$ and $\mathbf{h}_\alpha$ directly. By making use of relations (2.14), these equations can be represented in the form [10, 41]

$$\frac{d\rho_\alpha \mathbf{w}_\alpha}{dt} + \left(\nabla p_\alpha + \rho_\alpha \frac{d\mathbf{u}}{dt} - n_\alpha \mathbf{F}_\alpha\right) + \nabla\widehat{\boldsymbol{\pi}}_\alpha = m_\alpha \gamma_\alpha^{-1/2}\mathbf{R}_\alpha^{10} \tag{3.3}$$

$$\frac{d\pi_{\alpha ik}}{dt} + 2p_\alpha \varepsilon_{ik} + \frac{4}{5}\left\{\frac{\partial q_{\alpha i}}{\partial x_k}\right\} = kTR_{\alpha ik}^{20} \tag{3.4}$$

$$\frac{d\mathbf{h}_\alpha}{dt} + \frac{5}{2}\frac{k}{m_\alpha}p_\alpha \nabla T + \frac{kT}{m_\alpha}\nabla\hat{\boldsymbol{\pi}}_\alpha = \frac{kT}{2}\gamma_\alpha^{-1/2}\mathbf{R}_\alpha^{11} \tag{3.5}$$

The abbreviated notation

$$\{A_i B_k\} = \frac{1}{2}(A_i B_k + B_i A_k) - \frac{1}{3}A_l B_l \delta_{ik}, \quad \varepsilon_{ik} = \left\{\frac{\partial u_i}{\partial x_k}\right\}$$

has been used here.

In this case, $(\nabla\hat{\boldsymbol{\pi}}_\alpha)_i = \partial\pi_{\alpha ik}/\partial x_k$.

The specific expressions for the right-hand sides of Eqs (3.3)–(3.5) follow from the general representation of the quantities R_α^{mn} (2.29) and the expressions for the integral brackets of Sonine polynomials which associate them with the so-called Ω-integrals [1–3]. At the same time, the general expressions for the right-hand sides in the approximation of 13N moments have the form [10, 41–43]

$$m_\alpha \gamma_\alpha^{-1/2}\mathbf{R}_\alpha^{10} = \sum_\beta B_{\alpha\beta}^{(1)}(\mathbf{w}_\alpha - \mathbf{w}_\beta) + \sum \mu_{\alpha\beta}B_{\alpha\beta}^{(2)}\left(\frac{\mathbf{h}_\alpha}{m_\alpha p_\alpha} - \frac{\mathbf{h}_\beta}{m_\beta p_\beta}\right) \tag{3.6}$$

$$kTR_{\alpha ik}^{20} = \sum \frac{kT}{m_\alpha + m_\beta}\left(B_{\alpha\beta}^{(3)}\frac{\pi_{\alpha ik}}{p_\alpha} + B_{\alpha\beta}^{(4)}\frac{\pi_{\beta ik}}{p_\beta}\right) \tag{3.7}$$

$$\frac{kT}{2}\gamma_\alpha^{-1/2}\mathbf{R}_\alpha^{11} = \frac{kT}{m_\alpha}\sum\left[B_{\alpha\beta}^{(5)}\frac{\mathbf{h}_\alpha}{p_\alpha} + B_{\alpha\beta}^{(6)}\frac{\mathbf{h}_\beta}{p_\beta} + \frac{5}{2}\frac{\mu_{\alpha\beta}}{m_\alpha}B_{\alpha\beta}^{(2)}(\mathbf{w}_\alpha - \mathbf{w}_\beta)\right] \tag{3.8}$$

Here, $\mu_{\alpha\beta} = m_\alpha m_\beta/(m_\alpha + m_\beta)$ is the reduce mass of particles of species α and β and the coefficients $B_{\alpha\beta}^{(n)}$ are found to be linear functions of the Chapman – Cowling Ω-integrals [1]. In particular,

$$\begin{aligned} B_{\alpha\beta}^{(1)} &= -\frac{16}{3}n_\alpha n_\beta \mu_{\alpha\beta}\Omega_{\alpha\beta}^{11} \\ B_{\alpha\beta}^{(2)} &= -\frac{16}{3}n_\alpha n_\beta \mu_{\alpha\beta}\left(\frac{2}{5}\Omega_{\alpha\beta}^{12} - \Omega_{\alpha\beta}^{11}\right) \end{aligned} \tag{3.9}$$

Here,

$$\Omega_{\alpha\beta}^{lr} = \left(\frac{2\pi kT}{\mu_{\alpha\beta}}\right)^{1/2}\int_0^\infty\int_0^\pi \varsigma^{2r+3}e^{-\varsigma^2}(1-\cos^l\chi)\sigma_{\alpha\beta}(\varsigma,\chi)\sin\chi\, d\chi\, d\varsigma \tag{3.10}$$

Expressions for $B_{\alpha\beta}^{(n)}$ when $n = 3, 4, 5, 6$ have been presented earlier in [10, 43].

It is useful to carry out some estimates for the further simplification of the equations of moments. The quantity $B_{\alpha\beta}^{(1)}$ can be represented in the form $B_{\alpha\beta}^{(1)} = -n_\alpha \mu_{\alpha\beta}\tau_{\alpha\beta}^{-1}$, where $\tau_{\alpha\beta}^{-1} = (16/3)n_\beta \Omega_{\alpha\beta}^{11}$ is a certain effective collision frequency for particles of species α and β, since $\Omega_{\alpha\beta}^{11} = \langle g_{\alpha\beta}\rangle Q_{\alpha\beta}$, since $\langle g_{\alpha\beta}\rangle = (8kT/\pi\mu_{\alpha\beta})^{1/2}$ is the mean relative velocity of the particles and $Q_{\alpha\beta}$ is the effective diffusion scattering cross-section (for the model of molecules as solid, elastic spheres identical with the geometric collision cross-section). Specific calculations show that all the remaining coefficients $B_{\alpha\beta}^{(n)}$ are of the same order of magnitude as the coefficient $B_{\alpha\beta}^{(1)}$.

We will now adopt definite conditions for the macroscopic (hydrodynamic) description of the gas mixture and assume that the macroscopic parameters of the components in the mixture on the whole change slightly at distances of the order of the mean free path λ and during a time of the order of the time τ between collisions of the particles, that is,

$$\lambda/L_0 \ll 1, \quad \tau/\tau_0 \ll 1 \tag{3.11}$$

where L_0 and τ_0 are the characteristic linear and time scales of change in the macroscopic quantities. Note that conditions (3.11) correspond to the smallness of the Knudsen number, which is usually assumed in the CEM [1–3].

Taking into account the order of magnitude of the coefficients $B_{\alpha\beta}^{(n)}$, it is easy to see that, by virtue of the second condition of (3.11), the derivatives of the corresponding moments with respect to time on the left-hand sides of Eqs (3.3)–(3.5) can be neglected compared with the right-hand sides. In practice, this means that, in the case being considered of a slow change in the parameters of the mixture with time during a time interval equal to certain times between collisions of the particles, there is an onset of a quasi-equilibrium to which a system of equations which does not contain time derivatives of the moments of the distribution function can be applied approximately instead of Eqs (3.3–3.5).

As a rule, terms of the form $\nabla\hat{\boldsymbol{\pi}}_\alpha$ and $\partial q_{\alpha i}/\partial x_k$ in Eqs (3.3)–(3.5) are also of the order of $(\lambda/L_0)^2$ and taking them into account in the CEM corresponds to the Burnett approximation [1, 3]. It has been shown in [41] that the need to take account of the first of these can arise in the treatment of the diffusion of the components of a mixture in the case of slow flows (in the case of the steady viscous flow of a mixture, for example) when the longitudinal and transverse scales in the change in the gas parameters in the flow are quite different and the derivative of the viscous stress tensor with respect to the transverse coordinate is of the order of magnitude of the longitudinal pressure gradient. We shall return to this question in the next section.

4. THE EQUATIONS FOR THE MASS TRANSFER OF THE COMPONENTS IN STEFAN-MAXWELL FORM

We will now show that it is possible to obtain important relations from Eqs (3.3) for determining the diffusion velocities of the components of the mixture in a form which corresponds to the"forces in terms of flows" representation discussed in Section 1. Omitting, in accordance with conditions (3.11), the term

$d\rho_\alpha \mathbf{w}_\alpha/dt$ on the left-hand side of Eq. (3.3), instead of $d\mathbf{u}/dt$, we substitute an expression which follows from the general equation of motion of the mixture (2.37). As a result, instead of the left-hand side, we obtain

$$(\nabla p_\alpha - c_\alpha \nabla p) + \left(n_\alpha \mathbf{F}_\alpha - c_\alpha \sum_\alpha n_\alpha \mathbf{F}_\alpha\right) + (\nabla \hat{\pi}_\alpha - c_\alpha \nabla \hat{\pi})$$

where $c_\alpha = \rho_\alpha/\rho$ is the relative mass concentration of particles of species α.

We shall first consider the case when terms with derivatives of the viscous stress tensor can be neglected, assuming them to be small. The small of equations (3.3) then takes the form [10, 41]

$$p\mathbf{d}_\alpha = \sum_\beta B^{(1)}_{\alpha\beta}(\mathbf{w}_\alpha - \mathbf{w}_\beta) + \sum \mu_{\alpha\beta} B^{(2)}_{\alpha\beta}\left(\frac{\mathbf{h}_\alpha}{m_\alpha p_\alpha} - \frac{\mathbf{h}_\beta}{m_\beta p_\beta}\right) \tag{4.1}$$

where we have introduced the vector $\mathbf{d}_\alpha$, called the diffusion force vector (or, in the terminology of non-equilibrium thermodynamics, the "the thermodynamic diffusion force" [47]) defined as [2, 3]

$$\mathbf{d}_\alpha = \nabla x_\alpha + (x_\alpha - c_\alpha)\nabla \ln p - p^{-1}\left(n_\alpha \mathbf{F}_\alpha - c_\alpha \sum_{\beta=1}^{N} n_\beta \mathbf{F}_\beta\right) \tag{4.2}$$

Here, $x_\alpha = n_\alpha/n$ is the relative molar concentration of particles of species α.

Another important case corresponds to diffusion accompanying the steady viscous flow of a gas mixture in a channel [41], when it is possible to put $d\mathbf{u}/dt = 0$ in the equation of motion (2.37). As a result, this equation takes the form

$$\nabla p + \nabla \hat{\pi} - \sum_\alpha n_\alpha \mathbf{F}_\alpha = 0$$

and the left-hand side of Eq. (3.3) becomes

$$\nabla p_\alpha + \nabla \hat{\pi}_\alpha - n_\alpha \mathbf{F}_\alpha$$

It will be shown below that the solution of the equations of moments for the tensor expansion coefficients gives

$$\pi_{\alpha ik} = -2\eta_\alpha \varepsilon_{ik}, \quad \pi_{ik} = \sum_\alpha \pi_{\alpha ik} = -2\eta \varepsilon_{ik} \tag{4.3}$$

where η_α and η are the partial and total coefficients of viscosity. In this case, assuming that η_α and η are constant (or depend weakly on a coordinate), we obtain

$$\nabla \hat{\pi}_\alpha = (\eta_\alpha/\eta)\nabla \hat{\pi} = -(\eta_\alpha/\eta)\left(\nabla p - \sum_\alpha n_\alpha \mathbf{F}_\alpha\right)$$

As a result, the system of equations for the transport of components in the steady viscous flow of gas becomes

$$p\mathbf{d}^{\upsilon}_\alpha = \sum_\beta B^{(1)}_{\alpha\beta}(\mathbf{w}_\alpha - \mathbf{w}_\beta) + \sum \mu_{\alpha\beta} B^{(2)}_{\alpha\beta}\left(\frac{\mathbf{h}_\alpha}{m_\alpha p_\alpha} - \frac{\mathbf{h}_\beta}{m_\beta p_\beta}\right) \tag{4.4}$$

$$\mathbf{d}^{\upsilon}_\alpha = \nabla x_\alpha + \left(x_\alpha - \frac{\eta_\alpha}{\eta}\right)\nabla \ln p - p^{-1}\left(n_\alpha \mathbf{F}_\alpha - \frac{\eta_\alpha}{\eta}\sum_{\beta=1}^{N} n_\beta \mathbf{F}_\beta\right) \tag{4.5}$$

The main effect which arises in this case consists of the overdetermination of the coefficient of barodiffusion and the force term in the equations of the diffusions velocity of the components [41], since the quantity c_α in definition (4.2) is replaced by the ratio η_α/η. As a result, the barodiffusion ratio, unlike in the conventional case, becomes mainly a kinetic quantity and depends not only on the differences in the masses of the particles of the components but also on the differences in the effective collision cross-sections of particles of a different species [41].

The physical meaning of the equations for the diffusion velocities of the components becomes more obvious if it is noted that Eqs (4.1) or (4.4) can be obtained directly from the equation of motion of a separate component of the mixture, which is obtained by multiplying the complete (non-linearized) kinetic equation by $\psi_{\alpha i} = m_\alpha \upsilon_{\alpha i}$ and integrating with respect to the velocities. This equations can be represented in the form [10]

$$\begin{gathered} \rho_\alpha \frac{d_\alpha \mathbf{u}_\alpha}{dt} + \nabla \hat{\mathbf{P}}^*_\alpha - n_\alpha \mathbf{F}_\alpha = \mathbf{R}_\alpha \\ d_\alpha/dt = \partial/\partial t + \mathbf{u}_{\alpha l}\partial/\partial x_l \quad (\nabla \mathbf{P}^*_\alpha)_i = \partial p^*_{\alpha ik}/\partial x_k \end{gathered} \tag{4.6}$$

The quantity $\mathbf{P}^*_{\alpha ik}$ is connected with $\mathbf{P}_{\alpha ik}$ by the relation

$$\mathbf{P}^*_{\alpha ik} = \mathbf{P}_{\alpha ik} - \rho_\alpha w_{\alpha i} w_{\alpha k}, \quad \mathbf{P}_{\alpha ik} = p_\alpha \delta_{ik} + \pi_{\alpha ik}$$

In practice, the difference between $\mathbf{P}^*_{\alpha ik}$ and $\mathbf{P}_{\alpha ik}$ turns out to be unimportant since neglect of the quadratic terms with respect to the rates of diffusion corresponds to neglecting terms of the order of $(\lambda/L_0)^2$. The quantity $\mathbf{R}_\alpha = m_\alpha \gamma_\alpha^{-1/2} \mathbf{R}^{10}_\alpha$ is the magnitude of the average momentum transferred during collisions of particles of species α with all the particles of the remaining species in the mixture. It is also called the "diffusive force of friction" [10]. In the first term on the left-hand side of Eq. (4.6), the term $d_\alpha \mathbf{u}_\alpha/dt$ can be replaced by $d\mathbf{u}/dt$, which corresponds to neglecting terms of the order of $d\mathbf{w}_\alpha/dt$ compared with the terms on the right-hand side, which are of the order of magnitude $\tau^{-1}_{\alpha\beta}\mathbf{w}_\alpha$ and agrees with condition (3.11). As a result, we arrive at the initial equations for the diffusion velocities (3.3), in which it is necessary to omit the term with the time derivative of $\rho_\alpha \mathbf{w}_\alpha$, and, consequently, at Eqs (4.1) or (4.5).

The coefficients $B^{(1)}_{\alpha\beta}$ and $B^{(2)}_{\alpha\beta}$ on the right-hand side of Eq. (4.1) can be expressed using the coefficient of binary diffusion $[D_{\alpha\beta}]_1$, corresponding to the first approximation in the expansion in Sonine polynomials in the CCM [1–3], and the coefficient $C^*_{\alpha\beta}$

$$[D_{\alpha\beta}]_1 = \frac{3kT}{16n\mu_{\alpha\beta}\Omega^{11}_{\alpha\beta}}, \quad C^*_{\alpha\beta} = \frac{\Omega^{12}_{\alpha\beta}}{3\Omega^{11}_{\alpha\beta}} \tag{4.7}$$

As a result, Eqs (4.1) become

$$\sum_\beta \frac{n_\alpha n_\beta kT}{n[D_{\alpha\beta}]_1}(\mathbf{w}_\alpha - \mathbf{w}_\beta) = -p\mathbf{d}_\alpha + \sum_\beta \xi_{\alpha\beta}\left(\frac{\mathbf{h}_\beta}{m_\beta n_\beta} - \frac{\mathbf{h}_\alpha}{m_\alpha n_\alpha}\right) \tag{4.8}$$

Here,

$$\xi_{\alpha\beta} = \frac{n_\alpha n_\beta}{n[D_{\alpha\beta}]_1}\mu_{\alpha\beta}\left(\frac{6}{5}C^*_{\alpha\beta} - 1\right), \quad \beta \neq \alpha \tag{4.9}$$

Taking into account the second term on the right-hand side of Eq. (4.8) (after substituting the expressions for the partial reduced heat fluxes into it) we obtain the contribution corresponding to thermal diffusion and to the correction of the second approximation (with respect to the number of Sonine polynomials which are taken into account in the expansion in the CCM) to the coefficients of binary diffusion. If $\xi_{\alpha\beta} = 0$ (which holds, in particular, in the case of the model of Maxwellian molecules participating in collisions), the equations for determining the diffusion velocities of the components take the form

$$\sum_\beta \frac{n_\alpha n_\beta kT}{n[D_{\alpha\beta}]_1}(\mathbf{w}_\alpha - \mathbf{w}_\beta) = -p\mathbf{d}_\alpha \tag{4.10}$$

which corresponds to the usual representation of the equations of transport of components in a multicomponent mixture in Stefan–Maxwell form. Equations (4.10), as regards the accuracy of the calculation of the transport coefficients, correspond to the first approximation of the CCM. The equations, obtained by substituting the expressions for the partial reduced heat fluxes $\mathbf{h}_\alpha$ and $\mathbf{h}_\beta$ into the right-hand side of Eq. (4.8), correspond to the next (second) approximation of the CCM.

5. THE HEAT FLUX AND VISCOUS STRESS TENSOR

The partial reduced heat fluxes are found from the solution of Eqs (3.5) in which we omit the time derivative $d\mathbf{h}_\alpha/dt$ and the derivative with respect to the coordinate $d\pi_{\alpha ik}/\partial x_k$. In the viscous flow of a mixture, taking this last term into account leads to an additional contribution to the total heat flux, which is proportional to the pressure gradient and, also, to the second-order approximation, corrections to the barodiffusion constant in the equations for the diffusion velocities of the components [41, 10].

The system of equations obtained here, taking into account expression (3.8) and the form of the coefficients $B^{(n)}_{\alpha\beta}$ when $n = 5, 6, 7$ [10, 41, 43], can be written in the form

$$\frac{p^2}{T}\sum_{\beta=1}^{N}\Lambda^{11}_{\alpha\beta}\frac{\mathbf{h}_\beta}{p_\beta} = -\frac{p_\alpha}{T}\nabla T - \sum_{\beta\neq\alpha}\frac{kT}{m_\alpha}\xi_{\alpha\beta}(\mathbf{w}_\alpha - \mathbf{w}_\beta) \tag{5.1}$$

The coefficients $\Lambda^{11}_{\alpha\beta}$ are defined as [3, 10]

$$\Lambda^{11}_{\alpha\alpha} = \frac{x_\alpha^2}{[\lambda_{\alpha\alpha}]_1} + \frac{4}{25}\frac{T}{p}\sum_{\beta\neq\alpha}\frac{x_\alpha x_\beta}{(m_\alpha + m_\beta)^2 n[D_{\alpha\beta}]_1}\times$$

$$\times\left(\frac{15}{2}m_\alpha^2 + \frac{25}{4}m_\beta^2 + 3m_\beta^2 B^*_{\alpha\beta} + 4m_\alpha m_\beta A^*_{\alpha\beta}\right) \tag{5.2}$$

$$\Lambda^{11}_{\alpha\beta} = -\frac{4}{25}\frac{T}{p}\frac{m_\alpha m_\beta}{(m_\alpha + m_\beta)^2}\frac{x_\alpha x_\beta}{n[D_{\alpha\beta}]_1}\left(\frac{55}{4} - 3B^*_{\alpha\beta} - 4A^*_{\alpha\beta}\right),\quad \beta\neq\alpha$$

Here

$$[\lambda_{\alpha\alpha}]_1 = \frac{75}{32}\frac{k^2 T}{m_\alpha\Omega^{22}_{\alpha\alpha}},\quad A^*_{\alpha\beta} = \frac{\Omega^{22}_{\alpha\beta}}{2\Omega^{11}_{\alpha\beta}},\quad B^*_{\alpha\beta} = \frac{5\Omega^{12}_{\alpha\beta} - \Omega^{13}_{\alpha\beta}}{3\Omega^{11}_{\alpha\beta}} \tag{5.3}$$

$[\lambda_{\alpha\alpha}]_1$ is the thermal conductivity of a pure gas formed from particles of species α, calculated in the first approximation of the CCM [1–3] and $[D_{\alpha\beta}]_1$ is the binary coefficient of diffusion of particles of species α and β, defined by expression (4.7).

The solution of Eqs (5.1) can be represented in the form

$$\mathbf{h}_\alpha = -\lambda_\alpha\nabla T - \frac{kT^2}{p}\sum_{\beta=1}^{N}\sum_{\gamma=1}^{N}\frac{x_\alpha}{m_\beta}\xi_{\beta\gamma}\frac{|\Lambda^{11}|_{\beta\alpha}}{|\Lambda^{11}|}(\mathbf{w}_\beta - \mathbf{w}_\gamma) \tag{5.4}$$

Here

$$\lambda_\alpha = x_\alpha\sum_{\beta=1}^{N}x_\beta\frac{|\Lambda^{11}|_{\beta\alpha}}{|\Lambda^{11}|} \tag{5.5}$$

is the partial thermal conductivity. The notation $|A|$ and $|A|_{\alpha\beta}$ corresponds to an N-th-order determinant composed of the coefficients $A_{\alpha\beta}$ and the cofactor of the element $\beta\alpha$ of the determinant.

In accordance with relation (2.15) for $\mathbf{h}_\alpha$ and as a result of summation over α, the total heat flux in the mixture is given by the expression

$$\mathbf{q} = \frac{5}{2}\sum_{\alpha=1}^{N}p_\alpha\mathbf{w}_\alpha + \sum_{\alpha=1}^{N}\mathbf{h}_\alpha \tag{5.6}$$

or, taking definition (5.4) into account, by the expression

$$\mathbf{q} = \frac{5}{2}\sum_{\alpha=1}^{N} p_\alpha \mathbf{w}_\alpha - \lambda \nabla T - \frac{kT^2}{p}\sum_{\alpha=1}^{N}\sum_{\beta=1}^{N}\sum_{\gamma=1}^{N}\frac{x_\alpha}{m_\beta}\xi_{\beta\gamma}\frac{|\Lambda^{11}|_{\beta\alpha}}{|\Lambda^{11}|}(\mathbf{w}_\beta - \mathbf{w}_\gamma) \tag{5.7}$$

The last term on the right-hand side can be transformed by initially interchanging, under the summation signs, the subscripts α and γ and, then interchanging the subscripts β and α in the resulting expression. As a result, the expression for the total thermal flux in the mixture becomes

$$\mathbf{q} = \frac{5}{2}\sum_{\alpha=1}^{N} p_\alpha \mathbf{w}_\alpha - \lambda \nabla T - \sum_{\alpha=1}^{N}\sum_{\beta=1}^{N}\frac{n_\beta kT}{m_\alpha n[D_{\alpha\beta}]_1} D^T_{\alpha\beta}(\mathbf{w}_\alpha - \mathbf{w}_\beta) \tag{5.8}$$

Here

$$\lambda = \sum_{\alpha=1}^{N}\lambda_\alpha, \quad D^T_{\alpha\beta} = -\frac{\mu_{\alpha\beta}}{k}\left(\frac{6}{5}C^*_{\alpha\beta} - 1\right)\lambda_\alpha \tag{5.9}$$

where λ is the overall thermal conductivity of the mixture and $D^T_{\alpha\beta}$ is the coefficient of thermal diffusion of particles of species α and β [10, 41].

The expression for thermal conductivity can be represented in the form of the ratio of two determinants of order $N + 1$ and N

$$\lambda = -\frac{1}{|\Lambda^{11}|}\begin{vmatrix} \Lambda^{11}_{11} & \cdots & \Lambda^{11}_{1N} & x_1 \\ & & \vdots & \\ \Lambda^{11}_{N1} & \cdots & \Lambda^{11}_{NN} & x_N \\ x_1 & \cdots & x_N & 0 \end{vmatrix} \tag{5.10}$$

The expressions for λ and $D^T_{\alpha\beta}$ correspond in this case to the results of the full second approximation of the CCM [2, 3]. They are distinguished by the simpler form of the corresponding quantities which are presented, for example, in [2], where the thermal conductivity λ and thermal diffusion coefficient D^T_α (not to be confused with $D^T_{\alpha\beta}$) are expressed in terms of the ratio of $(2N + 1)$-th- and $2N$-th-order determinants, while the expressions obtained above include the ratios of $(N + 1)$-th- and N-th-order determinants. This is connected with the fact that those terms of the expansion in Sonine polynomials were used [2] when calculating the transport coefficients, which make only the first (non-vanishing) contribution to these coefficients. In particular, in order to obtain a non-zero result for the diffusion coefficient, it suffices just to take account of one term in the expansion (corresponding to the coefficient a^{10}_α in the scheme proposed here). The correct calculation of the λ and $D^T_{\alpha\beta}$ involves taking account of two coefficients (a^{10}_α and a^{11}_α in the scheme being considered).

Expressions for the partial and total viscous stress tensors have already been considered above. They follow from the solution of Eq. (3.4) in which the time derivative $d\pi_{\alpha ik}/dt$ and the derivative of the heat flux with respect to a coordinate are omitted. When the expression for $R^{20}_{\alpha ik}$ (3.7) is taken into account, these equations can be represented in the form

$$p^2\sum_{\beta=1}^{N} H^{00}_{\alpha\beta}\frac{\pi_{\beta ik}}{p_\beta} = -2p_0\xi_{ik} \tag{5.11}$$

where

$$H^{00}_{\alpha\alpha} = \frac{x^2_\alpha}{[\eta_{\alpha\alpha}]_1} + \sum_{\beta\neq\alpha}\frac{2x_\alpha x_\beta}{(m_\alpha + m_\beta)n[D_{\alpha\beta}]_1}\left(1 + \frac{3}{5}\frac{m_\beta}{m_\alpha}A^*_{\alpha\beta}\right)$$

$$H^{00}_{\alpha\beta} = -\frac{2x_\alpha x_\beta}{(m_\alpha + m_\beta)n[D_{\alpha\beta}]_1}\left(1 - \frac{3}{5}A^*_{\alpha\beta}\right), \quad \beta \neq \alpha \tag{5.12}$$

$$[\eta_{\alpha\alpha}]_1 = \frac{5}{8}\frac{kT}{\Omega^{22}_{\alpha\alpha}}$$

The coefficients $H^{00}_{\alpha\beta}$ are determined as previously in [2, 3] and $[\eta_{\alpha\alpha}]_1$ is the viscosity of the pure gas formed from particles of species α, calculated in the first approximation of the CCM [2, 3].

The linear relation for the components of the tensor of the viscous stresses in the mixture has the form

$$\pi_{ik} = -2\eta\varepsilon_{ik} \tag{5.13}$$

Here

$$\eta = \sum_{\alpha=1}^{N}\eta_\alpha = \sum_{\alpha=1}^{N}\sum_{\beta=1}^{N} x_\alpha x_\beta \frac{|H|_{\beta\alpha}}{|H|} \tag{5.14}$$

or

$$\eta = -\frac{1}{|H^{00}|}\begin{vmatrix} H^{00}_{11} & \cdots & H^{00}_{1N} & x_1 \\ \vdots & & \vdots & \vdots \\ H^{00}_{N1} & \cdots & H^{00}_{NN} & x_N \\ x_1 & \cdots & x_N & 0 \end{vmatrix} \tag{5.15}$$

The expression obtained is in complete agreement with the result corresponding to the first approximation of the CCM [2, 3].

We will not give here the corrections of the second approximation in the equations for the mass transport of the components which arise when the partial reduced heat fluxes (5.4) are substituted into Eqs (4.1) or (4.4). The corresponding results, together with the expressions for the heat fluxes and the viscous stress tensor, are considered in the following section within the framework of the higher approximations with respect to the number of Sonine polynomials in the expansion of the distribution function.

6. HIGHER APPROXIMATIONS

The higher approximations of the moment method correspond to taking a greater number of moments into account in the expansion of the distribution function (2.8) than the set of quantities considered above corresponding to the 13N moment approximation (3.1). The additional coefficients a^{mn}_α (when $m = 1, n > 1$ and, when $m = 2, n > 0$) are associated with moment of the distribution function which now do not have an explicit physical meaning. However, at each new stage of the approximation, when we confine ourselves to a finite number of terms in the expansion, the equations for these coefficients occur in the complete closed system of equations together with the equations for $a^{10}_{\alpha i}$, $a^{20}_{\alpha ik}$, and $a^{11}_{\alpha i}$ or $\mathbf{w}_\alpha$, $\pi_{\alpha ik}$ and $\mathbf{h}_\alpha$, and thereby refine the definition of these quantities at each order of approximation. Actually, the solution of this system enables one to find expressions for the diffusion velocities of interest, the heat fluxes and the viscous stress tensor, taking account of the contribution from the higher-order coefficients in the expansion.

Comparison with the expansions of the distribution function which are used in the CEM [1–3] shows that, at the level of the first approximation with respect to the Knudsen number λ/L_0, the equivalent expansion for f_α in the moment method must be chosen in the form

$$f_\alpha = f^{(0)}_\alpha\left(1 + \sum_{n=0}^{\infty}\sigma_{1n}a^{1n}_{\alpha i}H^{1n}_{\alpha i}(\xi_\alpha) + \sum_{n=0}^{\infty}\sigma_{2n}a^{2n}_{\alpha ik}H^{2n}_{\alpha ik}(\xi_\alpha)\right) \tag{6.1}$$

that is, the tensorial polynomials $P^{(m)}$ of not higher than the second rank ($m < 3$) must be included. The expansion for ϕ_α, taking account of the definition of the polynomials H^{mn}_α (2.9), can then be represented as

$$\phi_\alpha = \sum_{n=0}^{\infty} (-2)^n n! \left[\sigma_{1n} a^{1n}_{\alpha i} S^n_{3/2}\left(\frac{1}{2}\xi^2_\alpha\right)\xi_i + \sigma_{2n} a^{2n}_{\alpha ik} S^n_{5/2}\left(\frac{1}{2}\xi^2_\alpha\right)\left(\xi_i\xi_k - \frac{1}{3}\right)\xi^2\delta_{ik}\right] \tag{6.2}$$

The use of an arbitrary number of Sonine polynomials in expansion (6.2) enables one to calculate the fluxes and the corresponding transport coefficients in any given approximation. We note that, in order to establish complete correspondence between the Chapman – Enskog method and Grad's moment method at the level of the Burnett and following approximations to the distribution function, it can be proved to be necessary to take account of tensorial polynomials of a higher rank with respect to the index m, for example, the polynomials $P^{(3)}_{ijk}$. However, this question requires special discussion and is not considered here.

The system of algebraic equations for the coefficients a^{mn}_α follows from the general system of equation (3.23) if, in accordance with the analysis carried out above, the time derivatives and the spatial derivatives of the corresponding coefficients are omitted in each of the equations. At the same time, the time derivatives dn_α/dt, $d\mathbf{u}/dt$ and dT/dt on the left-hand side of the equations are replaced using the conservation equations (2.36)–(2.38), in which the derivatives of the dissipative fluxes are also omitted (that is, using Euler's equations). As a result, the left-hand sides of the equations for the scalar expansion coefficients ($m = 0$, $n = 0, 1$) vanish. When $m = 1$, we arrive at a system of equations for the vector quantities $\mathbf{a}^{1n}_{\alpha i}$, among which the coefficients $\mathbf{a}^{10}_{\alpha i}$ and $\mathbf{a}^{11}_{\alpha i}$ related to the quantities $\mathbf{w}_\alpha$ and $\mathbf{h}_\alpha$ of interest, appear

$$\begin{aligned}
&\sum_\beta \sum_{l=0}^{\xi-1} C^{10l}_{\alpha\beta}\mathbf{a}^{1l}_\beta = -\gamma^{-1/2}_\alpha n\mathbf{d}_\alpha \\
&\sum_\beta \sum_{l=0}^{\xi-1} C^{11l}_{\alpha\beta}\mathbf{a}^{1l}_\beta = -5\gamma^{-1/2}_\alpha n_\alpha \nabla \ln T \\
&\sum_\beta \sum_{l=0}^{\xi-1} C^{1nl}_{\alpha\beta}\mathbf{a}^{1l}_\beta = 0, \quad 1 < n \le \xi - 1
\end{aligned} \tag{6.3}$$

The quantity $\mathbf{d}_\alpha$ is defined by expression (4.2).

When $m = 2$, a system of equations is obtained for the coefficients $a^{2n}_{\alpha ik}$, among which the coefficient $a^{20}_{\alpha ik}$ associated with the partial viscous stress tensor $\pi_{\alpha ik}$, appears

$$\begin{aligned}
&\sum_\beta \sum_{l=0}^{\xi-1} C^{20l}_{\alpha\beta} a^{2l}_{\beta ik} = -2n_\alpha \varepsilon_{ik} \\
&\sum_\beta \sum_{l=0}^{\xi-1} C^{2nl}_{\alpha\beta} a^{2l}_{\beta ik} = 0, \quad 0 < n \le \xi - 1
\end{aligned} \tag{6.4}$$

The systems of vector and tensor linear algebraic equations (6.3) and (6.4) can be solved for the coefficients $\mathbf{a}^{1l}_\gamma$ and $a^{2l}_{\gamma ik}$ for any finite value of ξ. As a result, the coefficients $\mathbf{a}^{10}_\alpha$, $a^{20}_{\alpha ik}$ and $\mathbf{a}^{11}_\alpha$ or the parameters $\mathbf{w}_\alpha$, $\pi_{\alpha ik}$ and $\mathbf{h}_\alpha$ ("fluxes" in the terminology of the thermodynamics of non-equilibrium processes) are found. Note that the approximation of $13N$ moments, which has been considered above, corresponds to the use of the first two equations of (6.3) for the coefficients $a^{10}_{\alpha i}$ and $a^{11}_{\alpha i}$ ($\xi = 2$) and one equation of (6.4) for the coefficient $a^{20}_{\alpha ik}$ ($\xi = 1$).

Instead of the coefficients $C^{1nl}_{\alpha\beta}$, we introduce the symmetrical coefficients $q^{nl}_{\alpha\beta}$ which are defined as

$$q^{nl}_{\alpha\beta} = \frac{m^{1/2}_\alpha m^{1/2}_\beta}{nkT}(Q_{1nl})^{-1} C^{1nl}_{\alpha\beta} \tag{6.5}$$

or

$$q^{nl}_{\alpha\beta} = \frac{m^{1/2}_\alpha m^{1/2}_\beta}{nkT}[\delta_{\alpha\beta}\sum \hat{A}^{1nl}_{\alpha\gamma} + \hat{B}^{1nl}_{\alpha\beta}] \tag{6.6}$$

where

$$\hat{A}^{1nl}_{\alpha\beta} = n_\alpha n_\beta [S^l_{3/2}(W^2)\mathbf{W}, S^n_{3/2}(W^2)\mathbf{W}]'_{\alpha\beta}$$
$$\hat{B}^{1nl}_{\alpha\beta} = n_\alpha n_\beta [S^l_{3/2}(W^2)\mathbf{W}, S^n_{3/2}(W^2)\mathbf{W}]''_{\alpha\beta} \tag{6.7}$$

Taking definitions (2.14) into account system of equations (6.3) can be rewritten as

$$\sum_{\beta=1}^{N} q^{00}_{\alpha\beta}\mathbf{w}_\beta + \sum_{\beta=1}^{N} q^{01}_{\alpha\beta}\boldsymbol{\xi}_{\beta 1} + \sum_{\beta=1}^{N}\sum_{l=2}^{\xi-1} q^{0l}_{\alpha\beta}\boldsymbol{\xi}_{\beta l} = -\frac{3}{2}\mathbf{d}_\alpha \tag{6.8}$$

$$\sum_{\beta=1}^{N} q^{10}_{\alpha\beta}\mathbf{w}_\beta + \sum_{\beta=1}^{N} q^{11}_{\alpha\beta}\boldsymbol{\xi}_{\beta 1} + \sum_{\beta=1}^{N}\sum_{l=2}^{\xi-1} q^{1l}_{\alpha\beta}\boldsymbol{\xi}_{\beta l} = \frac{15}{4}x_\alpha \nabla \ln T \tag{6.9}$$

$$\sum_{\beta=1}^{N} q^{n0}_{\alpha\beta}\mathbf{w}_\beta + \sum_{\beta=1}^{N} q^{n1}_{\alpha\beta}\boldsymbol{\xi}_{\beta 1} + \sum_{\beta=1}^{N}\sum_{l=2}^{\xi-1} q^{nl}_{\alpha\beta}\boldsymbol{\xi}_{\beta l} = 0, \quad 1 < n \le \xi - 1 \tag{6.10}$$

This system of equations must be supplemented with the condition

$$\sum_{\alpha=1}^{N} n_\alpha m_\alpha \mathbf{w}_\alpha = 0 \tag{6.11}$$

The new variables

$$\boldsymbol{\xi}_{\beta l} = \frac{3}{2} Q_{1nl} \gamma_\beta^{-1/2} \mathbf{a}_\beta^{1l} \tag{6.12}$$

are used in Eqs (6.8)–(6.10) and, in particular,

$$\boldsymbol{\xi}_{\beta 0} = \mathbf{w}_\beta, \quad \boldsymbol{\xi}_{\beta 1} = -\frac{2}{5}\mathbf{h}_\beta = -\frac{2}{5}\left(\mathbf{q}_\beta - \frac{5}{2}p_\beta \mathbf{w}_\beta\right) \tag{6.13}$$

System of equations (6.8)–(6.10) is completely identical to the equations of the modified CCM obtained earlier [38–45]. In the case where $\xi = 2$, this system corresponds to the equations for the variables $\mathbf{w}_\alpha$ and $\mathbf{h}_\alpha$ obtained in Sections 4 and 5 in the $13N$ moments approximation.

In the general case, the solution of system of equations (6.8)–(6.10) leads to expressions for the diffusion flux $\mathbf{J}_\alpha = \rho_\alpha \mathbf{w}_\alpha$ and the reduced total heat flux $\mathbf{h} = \sum_{\alpha=1}^{N} \mathbf{h}_\alpha$, which depend linearly on the "thermodynamic forces" $\mathbf{d}_\beta$ and $\nabla \ln T$

$$\mathbf{J}_\alpha = -\rho_\alpha \sum_{\beta=1}^{N} D_{\alpha\beta}(\xi)\mathbf{d}_\beta - \rho_\alpha D_{T\alpha}(\xi)\nabla \ln T \tag{6.14}$$

$$\mathbf{h} = \mathbf{q} - \frac{5}{2}kT\sum_{\alpha=1}^{N} n_\alpha \mathbf{w}_\alpha = -\lambda'(\xi)\nabla T - p\sum_{\alpha=1}^{N} D_{T\alpha}\mathbf{d}_\alpha \tag{6.15}$$

Here $D_{\alpha\beta}$ and $D_{T\alpha}$ are the diffusion and thermal diffusion coefficients of a multicomponent mixture [2, 3] and λ' is the "instantaneous" thermal conductivity of the mixture (for the case when all the diffusion thermodynamic forces vanish).

Relations (6.14) and (6.15) correspond to well-known results of the standard CEM procedure [2, 3]. The coefficients of proportionality in these relations (the transport coefficients), determined in an arbitrary approximation with respect to ξ, are written in the form of ratios of $N\xi$ + 1-th- and $N\xi$-th-order determinants composed of the elements $q^{nl}_{\alpha\beta}$.

Another algorithm for solving system of equations (6.8)–(6.10) has been proposed [38, 4, 5] which leads to the equations for the diffusion velocities of the components being written down in Stefan–Maxwell form, and to an expression for the total heat flux, which depends linearly on the temperature gradient and the diffusion velocities of the components $\mathbf{w}_\beta$ (but not the diffusion forces $\mathbf{d}_\beta$). In the case of such an approach, system of equations (6.9)–(6.10), which can be represented as

$$\sum_{\beta=1}^{N}\sum_{l=1}^{\xi-1} q_{\alpha\beta}^{nl}\xi_{\beta l} = \frac{15}{4}x_\alpha \nabla \ln T\delta_{n1} - \sum_{\beta=1}^{N} q_{\alpha\beta}^{n0}\mathbf{w}_\beta \qquad (6.16)$$
$$\alpha = 1, \dots, N; \quad n = 1, \dots, \xi - 1$$

is initially solved.

The expressions for the reduced partial heat reflux are written in this case in the form

$$\mathbf{h}_\alpha = -\frac{5}{2}n_\alpha kT\xi_{\alpha 1} = -\lambda_\alpha \nabla T + nkT\sum_{\beta=1}^{N} k_{T\alpha\beta}\mathbf{w}_\beta \qquad (6.17)$$

where

$$\lambda_\alpha = -\frac{75}{8}\frac{nk}{|q|}x_\alpha \begin{vmatrix} 0 & 0 & \dots & \delta_{\alpha s} & 0 & \dots \\ x_r & q_{rs}^{11} & \dots & q_{rs}^{1n} & q_{rs}^{1,n+1} & \dots \\ 0 & q_{rs}^{21} & \dots & q_{rs}^{2n} & q_{rs}^{2,n+1} & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \end{vmatrix} \qquad (6.18)$$

$$k_{T\alpha\beta} = -\frac{5}{2}\frac{1}{|q|}x_\alpha \begin{vmatrix} 0 & 0 & \dots & \delta_{\alpha s} & 0 & \dots \\ q_{r\beta}^{10} & q_{rs}^{11} & \dots & q_{rs}^{1n} & q_{rs}^{1,n+1} & \dots \\ q_{r\beta}^{20} & q_{rs}^{21} & \dots & q_{rs}^{2n} & q_{rs}^{2,n+1} & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \end{vmatrix} \qquad (6.19)$$

Here, $q_{\alpha s}^{mn}$ denotes a square matrix of order N with elements $q_{\alpha\beta}^{mn}$, $\delta_{\alpha s}$ denotes the corresponding row of Kronecker deltas $\delta_{\alpha\beta}$, and x_r is a column of the values of x_α (α, $\beta = 1, \dots, N$). The notation $|q|$ corresponds to the determinant of system of equations (6.16) with elements q_{rs}^{mn}.

Note that the coefficients $q_{\alpha\beta}^{mn}$ satisfy the conditions [5]

$$q_{\alpha\beta}^{mn} = q_{\beta\alpha}^{mn}, \quad \sum_{\alpha=1}^{N} q_{\alpha\beta}^{0n} = 0, \quad \sum_{\varepsilon=1}^{N} q_{\alpha\beta}^{m0} = 0 \qquad (6.20)$$

The last two relations follow from the law of conservation of momentum of the mixture.

For the total reduced heat flux in any approximation with respect to ξ, we have

$$\mathbf{h} = -\lambda(\xi)\nabla T + nkT\sum_{\alpha=1}^{N} k_{T\alpha}(\xi)\mathbf{w}_\alpha \qquad (6.21)$$

The thermal conductivity λ and the thermal diffusion ratios $k_{T\alpha}$ are given by the expressions

$$\lambda(\xi) = \sum_{\alpha=1}^{N}\lambda_\alpha = -\frac{75}{8}\frac{nk}{|q|}\begin{vmatrix} 0 & x_s & 0 & \dots \\ x_r & q_{rs}^{11} & q_{rs}^{12} & \dots \\ 0 & q_{rs}^{21} & q_{rs}^{22} & \dots \\ \dots & \dots & \dots & \dots \end{vmatrix} \qquad (6.22)$$

$$k_{T\alpha}(\xi) = \sum_{\beta=1}^{N} k_{T\alpha\beta} = -\frac{5}{8}\frac{1}{|q|}\begin{vmatrix} 0 & x_s & 0 & \cdots \\ q^{10}_{r\alpha} & q^{11}_{rs} & q^{12}_{rs} & \cdots \\ q^{20}_{r\alpha} & q^{21}_{rs} & q^{22}_{rs} & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{vmatrix} \tag{6.23}$$

By virtue of conditions (6.20), the values of the thermal diffusion ratios $k_{T\alpha}$ are not independent but are connected by the relation

$$\sum_{\alpha=1}^{N} k_{T\alpha} = 0 \tag{6.24}$$

Taking definition (2.15) and condition (6.20) into account, the expression for the total heat flux in the mixture can be represented in the form

$$\mathbf{q} = \frac{5}{2}kT\sum_{\alpha=1}^{N} n_\alpha \mathbf{w}_\alpha - \lambda(\xi)\nabla T + nkT\sum_{\alpha=1}^{N} k_{T\alpha}(\xi)\mathbf{w}_\alpha \tag{6.25}$$

The Stefan–Maxwell equations are obtained by substituting the solutions of Eqs (6.9) and (6.10) into Eq. (6.8). The first term on the left in Eq. (6.8) is transformed using the expression for $q^{00}_{\alpha\beta}$ [2, 3]

$$q^{00}_{\alpha\beta} = \frac{3}{2}\delta_{\alpha\beta}\sum_{\gamma=1}^{N}\frac{x_\alpha x_\gamma}{[D_{\alpha\gamma}]_1} - \frac{3}{2}\frac{x_\alpha x_\beta}{[D_{\alpha\beta}]_1} \tag{6.26}$$

The final result is written in the form [40]

$$-p\mathbf{d}_\alpha = \sum_{\beta\neq\alpha}^{N}\frac{n_\alpha n_\beta kT}{n[D_{\alpha\beta}]_1 f_{\alpha\beta}}(\mathbf{w}_\alpha - \mathbf{w}_\beta) + pk_{T\alpha}\nabla\ln T \tag{6.27}$$

Here, $f_{\alpha\beta}(\xi) = [1 - D_{\alpha\beta}(\xi)]^{-1}$ is the correction of the higher approximation to the coefficient of binary diffusion $[D_{\alpha\beta}]_1$, where

$$\Delta_{\alpha\beta}(\xi) = \frac{2}{3}\frac{n^2[D_{\alpha\beta}]_1}{n_\alpha n_\beta |q|}\begin{vmatrix} 0 & q^{01}_{\beta s} & q^{02}_{\beta s} & \cdots \\ q^{10}_{r\alpha} & q^{11}_{rs} & q^{12}_{rs} & \cdots \\ q^{20}_{r\alpha} & q^{21}_{rs} & q^{22}_{rs} & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{vmatrix} \tag{6.28}$$

Unlike the usual expressions for the diffusion, fluxes and the heat flux (6.14) and (6.15), obtained within the framework of the CEM and which correspond to the "fluxes in terms of forces" representation, the diffusion equations in the Stefan–Maxwell form (6.27) and the expression for the heat flux (6.25) correspond to a system of transport equations which is solved for "forces in terms of fluxes" in any approximation with respect to ξ. The advantage of the expressions obtained using this approach lies in the fact that the transport coefficients in them are expressed in terms of a ratio of determinants of a lower order, in fact, a ratio of $(N(\xi - 1) + 1)$-th- and $(N(\xi - 1)$-th-order determinants, which appreciably simplifies the calculation of the corresponding quantities. Furthermore, the expression for q in the form of (6.25) is preferable in specific use since, unlike the "instantaneous" thermal conductivity λ', the "true" thermal conductivity λ can be measured directly. This is due to the fact that a steady state in the mixture corresponding to the experimental conditions is only established after the transport of a mass of a component due to the temperature gradient (thermal diffusion) is compensated by the mass diffusion flux due to the concentration gradient which has arisen. The total diffusion fluxes in this case vanish and the flux is given by expression (6.25), which corresponds to Fourier's law with a "true" thermal conductivity λ.

Note that the coefficients $q^{nl}_{\alpha\beta}$ are related to the known coefficients $\Lambda^{nl}_{\alpha\beta}$ used in [3] by the relation

$$\Lambda^{nl}_{\alpha\beta} = \frac{8}{75}\frac{T}{p}q^{nl}_{\alpha\beta} \tag{6.29}$$

We will now write the system of equations (6.8)–(6.9) for the case when $\xi = 2$, using relation (6.29)

$$\begin{aligned}
&\sum_{\beta=1}^{N}\Lambda^{00}_{\alpha\beta}\mathbf{w}_\beta - \frac{2}{5}\sum_{\beta=1}^{N}\Lambda^{01}_{\alpha\beta}\frac{\mathbf{h}_\beta}{p_\beta} = -\frac{4}{25}\frac{T}{p}\mathbf{d}_\alpha \\
&\sum_{\beta=1}^{N}\Lambda^{10}_{\alpha\beta}\mathbf{w}_\beta - \frac{2}{5}\sum_{\beta=1}^{N}\Lambda^{11}_{\alpha\beta}\frac{\mathbf{h}_\beta}{p_\beta} = \frac{2}{5}\frac{T}{p}x_\alpha\nabla\ln T
\end{aligned} \tag{6.30}$$

The coefficients $\Lambda^{nl}_{\alpha\beta}$ are defined by expressions (5.2). For the coefficients $\Lambda^{00}_{\alpha\beta}$ and $\Lambda^{01}_{\alpha\beta} = \Lambda^{10}_{\alpha\beta}$, we have [3]

$$\Lambda^{00}_{\alpha\alpha} = \frac{4}{25}\sum_{\beta\neq\alpha}\frac{x_\alpha x_\beta}{nk[D_{\alpha\beta}]_1}; \quad \Lambda^{00}_{\alpha\beta} = -\frac{4}{25}\frac{x_\alpha x_\beta}{nk[D_{\alpha\beta}]_1}, \quad \alpha\neq\beta \tag{6.31}$$

$$\begin{aligned}
\Lambda^{01}_{\alpha\alpha} &= -\frac{2}{5}\sum_{\beta\neq\alpha}\frac{x_\alpha x_\beta}{nk[D_{\alpha\beta}]_1}\frac{\mu_{\alpha\beta}}{m_\alpha}\left(\frac{6}{5}C^*_{\alpha\beta} - 1\right) \\
\Lambda^{10}_{\alpha\beta} &= -\frac{2}{5}\frac{x_\alpha x_\beta}{nk[D_{\alpha\beta}]_1}\frac{\mu_{\alpha\beta}}{m_\beta}\left(\frac{6}{5}C^*_{\alpha\beta} - 1\right), \quad \alpha\neq\beta
\end{aligned} \tag{6.32}$$

It is easily shown that Eqs (6.30) are completely equivalent to Eqs (4.1) and (5.1) obtained earlier in the approximation of 13N moments. Correspondingly, the expressions for the thermal conductivity and thermal diffusion coefficient are also identical. At the same time, the thermal diffusion ratio $k_{T\alpha}$ is related to the thermal diffusion coefficients of particles of species α and β, introduced in the preceding section, by the relations

$$\begin{aligned}
k_{T\alpha} &= \sum_{\beta=1}^{N}\frac{n_\alpha n_\beta}{n^2[D_{\alpha\beta}]_1}\left(\frac{D^T_{\alpha\beta}}{m_\alpha n_\alpha} - \frac{D^T_{\beta\alpha}}{m_\beta n_\beta}\right) = \\
&= -\frac{T}{p}\sum_{\beta=1}^{N}\frac{\mu_{\alpha\beta}n_\alpha n_\beta}{n^2[D_{\alpha\beta}]_1}\left(\frac{6}{5}C^*_{\alpha\beta} - 1\right)\left(\frac{\lambda_\alpha}{m_\alpha n_\alpha} - \frac{\lambda_\beta}{m_\beta n_\beta}\right)
\end{aligned} \tag{6.33}$$

We now consider the equations for determining the partial viscous stress tensors and transform the coefficients in Eqs (6.4) using the relation

$$\bar{q}^{nl}_{\alpha\beta} = \frac{1}{n^2}(Q_{2nl})^{-1}C^{2nl}_{\alpha\beta} \tag{6.34}$$

or

$$\bar{q}^{nl}_{\alpha\beta} = \frac{1}{n^2}(\delta_{\alpha\beta}\sum\tilde{A}^{2nl}_{\alpha\gamma} + \tilde{B}^{2nl}_{\alpha\beta}) \tag{6.35}$$

where

$$\begin{aligned}
\tilde{A}^{2nl}_{\alpha\beta} &= n_\alpha n_\beta\left[S^l_{5/2}(W^2)\left(W_iW_k - \frac{1}{3}\delta_{ik}W^2\right), S^n_{5/2}(W^2)\left(W_iW_k - \frac{1}{3}\delta_{ik}W^2\right)\right]'_{\alpha\beta} \\
\tilde{B}^{2nl}_{\alpha\beta} &= n_\alpha n_\beta\left[S^l_{5/2}(W^2)\left(W_iW_k - \frac{1}{3}\delta_{ik}W^2\right), S^n_{5/2}(W^2)\left(W_iW_k - \frac{1}{3}\delta_{ik}W^2\right)\right]''_{\alpha\beta}
\end{aligned} \tag{6.36}$$

Equations (6.4) are then rewritten as

$$\begin{gathered}\sum_{\beta=1}^{N}\bar{q}_{\alpha\beta}^{00}\sigma_{\beta ik}^{(0)}+\sum_{\beta=1}^{N}\sum_{l=1}^{\xi-1}\bar{q}_{\alpha\beta}^{0l}\sigma_{\beta ik}^{(l)} = -5x_\alpha\varepsilon_{ik}\\ \sum_{\beta=1}^{N}\bar{q}_{\alpha\beta}^{n0}\sigma_{\beta ik}^{(0)}+\sum_{\beta=1}^{N}\sum_{l=1}^{\xi-1}\bar{q}_{\alpha\beta}^{nl}\sigma_{\beta ik}^{(0)} = 0,\quad 0<n\le\xi-1\end{gathered}\tag{6.37}$$

where

$$\sigma_{\beta ik}^{(l)} = \frac{5}{2}nkTQ_{2n l}a_\beta^{2l}\tag{6.38}$$

The partial viscous stress tensor is related to $\sigma_{\alpha ik}^{(0)}$ by the relation $\pi_{\alpha ik} = x_\alpha\sigma_{\alpha ik}^{(0)}$. Solution of Eqs (6.37) gives

$$\pi_{\alpha ik} = -2\eta_\alpha\varepsilon_{ik},\quad \pi_{ik} = -2\eta\varepsilon_{ik}\tag{6.39}$$

where the partial and total coefficients of viscosity are determined by the expressions

$$\eta_\alpha(\xi) = -\frac{5}{2}\frac{1}{|q|}x_\alpha\begin{vmatrix}0 & 0 & \ldots & \delta_{\alpha s} & 0 & \ldots\\ x_r & \bar{q}_{rs}^{00} & \ldots & \bar{q}_{rs}^{0n} & \bar{q}_{rs}^{0,n+1} & \ldots\\ 0 & \bar{q}_{rs}^{10} & \ldots & \bar{q}_{rs}^{1n} & \bar{q}_{rs}^{1,n+1} & \ldots\\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots\end{vmatrix}\tag{6.40}$$

$$\eta(\xi) = \sum_{\alpha=1}^{N}\eta_\alpha = -\frac{5}{2}\frac{1}{|q|}\begin{vmatrix}0 & x_s & 0 & \ldots\\ x_r & \bar{q}_{rs}^{00} & \bar{q}_{rs}^{01} & \ldots\\ 0 & \bar{q}_{rs}^{10} & \bar{q}_{rs}^{11} & \ldots\\ \ldots & \ldots & \ldots & \ldots\end{vmatrix}\tag{6.41}$$

The coefficients $\bar{q}_{\alpha\beta}^{nl}$ are related to the well-known Ferziger–Kaper coefficients $H_{\alpha\beta}^{nl}$ [3] by the relation

$$\bar{q}_{\alpha\beta}^{nl} = \frac{5}{2}\frac{p}{kT}H_{\alpha\beta}^{nl}\tag{6.42}$$

In the case when $\xi = 1$, the equations for determining $\pi_{\alpha ik}$ take the form

$$\sum H_{\alpha\beta}^{00}\frac{\pi_{\beta ik}}{x_\beta} = -2x_\alpha\varepsilon_{ik}\tag{6.43}$$

which corresponds completely to Eqs (5.10) obtained in the approximation of $13N$ moments.

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